One-way full-waveform inversion using frequency-domain model extension

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SUMMARY

Conventional full-waveform inversion is notorious for being sensitive to the choice of a starting model. Moreover, it requires low-frequency data to be able to reconstruct the model accurately. To overcome these limitations, we propose a formulation of FWI using frequency-extended slowness and impedance models. We pose and solve this problem in the frequency domain using one-way wave extrapolation operators.

Extended slowness model allows for wavefields to be phase-shifted (equivalently, time-shifted) leading to non-physical wave propagation and forward scattering. Frequency-extended impedance model, used to compute reflectivity, allows for non-physical backward scattering and is updated simultaneously with slowness. As a result of iteratively updating the two extended models, FWI data residuals can be modeled accurately even in the presence of large errors in the starting velocity (slowness) model.

During the inversion, we use multi-scale approach in space by implementing variable spline grid parametrization. We utilize a fine spline grid for the extended impedance to capture the reflectivity of the model and to be able to accurately model the data residuals. We start the inversion with a coarse spline grid for the extended slowness, leading to low-wavenumber updates, and gradually increase it at each step of multi-scale iteration. The variability of the model parameters with frequency can be controlled in a similar way by choosing appropriate spline grid parametrization.

As a result, accurate slowness model can be recovered even from a rough starting model and with no low-frequency components in the data. The proposed method does not require intermediate step of solving a least-squares problem for extended reflectivity and is computationally attractive. We demonstrate the effectiveness of our approach on a synthetic example.

INTRODUCTION

Model extension has been demonstrated to be a powerful technique for dealing with the cycle-skipping problem of full-waveform inversion by various authors (Biondi and Almomin, 2012; Huang et al., 2017, Barnier et al., 2018, Barnier et al., 2023). Velocity model reconstruction based on this tool has been shown to result in better solutions compared to the conventional FWI (Almomin and Biondi, 2012; Barnier et al., 2019). Inversion methods such as tomographic full-waveform inversion (TFWI) or full-waveform inversion with model extension (FWIME) are based on the concept of finding and using an extended reflectivity as a means for data residuals matching in the presence of inaccurate starting velocity model. Using the reconstructed extended reflectivity avoids cycle-skipping, and as a result, tomographic component of the FWI-gradient can be computed accurately and used to update the velocity model. These methods rely on solving for extended reflectivity using extended least-squares migration at each iteration of the velocity update. Moreover, the extended scattering (imaging) condition requires a convolution (cross-correlation) of the background wavefield with the extended reflectivity (or scattered wavefield) in time or in space. Therefore, existing methods based on the extended models can be computationally unfeasible in realistic scenarios.

We propose using model extension in the frequency domain, that offers both computational and theoretical advantages. Using frequency-domain (complex-valued) slowness model leads to a modified (extended) wave equation. Solving this equation directly removes the necessity to introduce extended reflectivity and leads to a fully non-linear inversion scheme without the need to solve an inner least-squares problem.

Performing frequency-extended imaging condition is computationally efficient and involves correlating and keeping each frequency component of the wavefields independently. In this way, every iteration of the proposed extended FWI has the same computational cost as the conventional frequency-domain FWI. In order to converge to a physical solution, a regularization term is added to promote similarity in the extended model across frequencies.

To propagate wavefields efficiently in the frequency-domain we use one-way wave extrapolation operators (Claerbout, 1985) based on the phase-shift plus interpolation (Gazdag and Sguazzero, 1984) and split-step correction (Stoffa et al., 1990). Methods based on the one-way wave extrapolation for wave simulation have been suggested before (Berkhout, 2012, Verschuur et al., 2016), however, their adoption for the waveform inversion still remains limited. To be able to model reflected waves we introduce a variable complex-valued extended impedance model. We approximate the reflection operator as scaling of the downward going wavefields by the reflectivity equal to zero-incidence reflection coefficient.

The slowness-impedance parametrization of the forward modeling operator (and its linearized version) leads to natural separation of the forward- and backward-scattering components of the FWI gradients. We use different spline grid (Knott, 2000)s parametrization for the two models to control their respective smoothness. Having fine (high-resolution) spline grid for impedance allows to capture the extended reflectivity and preserves the property of data-residual matching. On the other hand, the slowness model is projected to a coarser, or low-resolution, spline grid to promote smooth updates at early stages of inversion. This strategy leads to correct low-wavenumber updates to the slowness model and allows to recover accurate models starting from a rough initial estimate and data not containing low frequencies.

METHODOLOGY

We propose using the following modification to the acoustic wave equation:

$$\int \mathbf{s}^2(\tau)\frac{\partial^2 \mathbf{p}}{\partial t^2}(\mathbf{x}, t-\tau)d\tau - \nabla^2 \mathbf{p}(\mathbf{x}, t) = 0, \tag{1}$$

where the second derivative of the pressure field $\mathbf{p}$ is convolved with the time-varying slowness-squared $\mathbf{s}^2(\tau)$, rather than being scaled by it as in the conventional wave equation. Applying Fourier transform along the time axis we get:

$$-\omega^2 \hat{\mathbf{s}}^2(\omega)\hat{\mathbf{p}}(\mathbf{x}, \omega) - \nabla^2 \hat{\mathbf{p}}(\mathbf{x}, \omega) = 0, \tag{2}$$

where convolution in time becomes multiplication in the frequency domain of the pressure second derivative $(-\omega^2)\hat{\mathbf{p}}(\mathbf{x}, \omega)$ and frequency-varying slowness-squared $\hat{\mathbf{s}}^2(\omega)$. Note, that the equation 2 is basically the Helmholtz equation with frequency-varying complex-valued slowness model.

Extended modeling using one-way approximation

To solve the wave equation 2 we apply Fourier transform along the lateral axis and factorize it into two one-way wave equations:

$$\left(\frac{\partial}{\partial z} - i\sqrt{\omega^2 \hat{\mathbf{s}}^2(\omega) - k_x^2}\right)\left(\frac{\partial}{\partial z} + i\sqrt{\omega^2 \hat{\mathbf{s}}^2(\omega) - k_x^2}\right) = 0. \tag{3}$$

We solve one-way wave equation using phase-shift plus interpolation (Gazdag and Sguazzero, 1984) and split-step correction (Stoffa et al., 1990). Similar to solving conventional one-way wave equation, the wavefields are extrapolated in depth but the slowness is different for each frequency component of the wavefield.

Equation 3 allows us to propagate wavefields upward and downward, however, it does not take into account reflections. We approximate reflection process by scaling the downgoing wavefield by zero-incidence reflectivity that can be computed using either slowness-only (constant density)

$$R_s(x, z_j, \omega) = \frac{s(x, z_j, \omega) - s(x, z_{j+1}, \omega)}{s(x, z_j, \omega) + s(x, z_{j+1}, \omega)} \tag{4}$$

or impedance parametrization

$$R_I(x, z_j, \omega) = \frac{I(x, z_j, \omega) - I(x, z_{j+1}, \omega)}{I(x, z_j, \omega) + I(x, z_{j+1}, \omega)} \tag{5}$$

The downward going wavefield is scaled by the reflectivity computed using equations above and re-injected as a source of upward propagating wavefield. Hence, the process of modeling reflected waves using proposed approach can be represented as a chain of non-linear operators in slowness-only:

$$\mathbf{f}(\mathbf{s}) = \mathbf{L}^T(\mathbf{s})\mathbf{R}(\mathbf{s})\mathbf{L}(\mathbf{s}) \tag{6}$$

or slowness-impedance parametrization

$$\mathbf{f}(\mathbf{s}, \mathbf{I}) = \mathbf{L}^T(\mathbf{s})\mathbf{R}(\mathbf{I})\mathbf{L}(\mathbf{s}), \tag{7}$$

where $\mathbf{L}$ represents downward continuation, $\mathbf{L}^T$ – upward continuation and $\mathbf{R}$ – reflection operators. Downward continuation operator can be represented as a lower triangular matrix acting on the wavefield, where each row represents an extrapolation operator to the next depth level. Note, that upward continuation is a transpose of a downward continuation (not adjoint) because the extrapolation proceeds forward in time starting from the bottom to the top of the model.

Extended full-waveform inversion

Using the extended forward modeling operator $\mathbf{f}(\mathbf{m})$, where $\mathbf{m}$ could be either represented by slowness (equation 6) or slowness-impedance pair (equation 7), we set up the inversion problem.

To compute the gradient with respect to the model parameters, we linearize the forward operator $\mathbf{f}(\mathbf{m})$, using the chain rule:

$$\frac{d\mathbf{f}}{d\mathbf{m}}(\mathbf{m}) = \frac{d\mathbf{U}}{d\mathbf{m}}(\mathbf{m})\mathbf{R}(\mathbf{m})\mathbf{D}(\mathbf{m}) + \mathbf{U}(\mathbf{m})\mathbf{R}(\mathbf{m})\frac{d\mathbf{D}}{d\mathbf{m}}(\mathbf{m}) + \mathbf{U}(\mathbf{m})\frac{d\mathbf{R}}{d\mathbf{m}}(\mathbf{m})\mathbf{D}(\mathbf{m}) \tag{8}$$

The first term represents forward scattering upward, second – forward scattering downward and the third – backward scattering components. The adjoint of the operator in the equation 8 is used to compute FWI gradient. Note, that slowness-impedance parametrization in equation 7 allows for natural separation between tomographic and reflectivity component of the FWI gradient that is used during the inversion.

To converge to a physical solution, where the slowness is the same for each frequency, we add a regularization term with the DSO as a regularization operator (Symes and Carazzone, 1991, Symes, 2008). By using the duality property of the Fourier transform $\mathscr{F}$, the frequency-domain equivalent of the time-lag DSO operator $\mathbf{D}_\omega$ becomes

$$\tau \overset{\mathscr{F}}{\Longrightarrow} i\frac{\partial}{\partial \omega}, \tag{9}$$

which has the meaning of promoting similarity of the extended model across the frequency axis by minimizing its first derivative.

Finally, thanks to the separation of the variables, we can use different spline parametrization for extended slowness and impedance models to control their relative smoothness:

$$\mathbf{m} = \begin{bmatrix}\mathbf{s}\\ \mathbf{I}\end{bmatrix} = \begin{bmatrix}\mathbf{S}_s & 0\\ 0 & \mathbf{S}_I\end{bmatrix}\begin{bmatrix}\mathbf{p}_s\\ \mathbf{p}_I\end{bmatrix} = \mathbf{S}\mathbf{p}, \tag{10}$$

where $\mathbf{p}_s$ represents the preconditioned variable for slowness, $\mathbf{p}_I$ – preconditioned variable for impedance, and operators $\mathbf{S}_s$ and $\mathbf{S}_I$ – spline interpolation operators for slowness and impedance respectively.

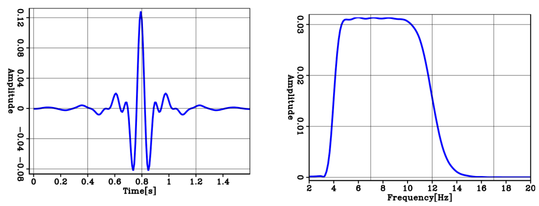
As a result, we can pose the problem of extended full-waveform inversion as minimization of the following objective function over a preconditioned model $\mathbf{p}$:

$$\phi(\mathbf{p}) = \frac{1}{2}\|\mathbf{K}\mathbf{f}(\mathbf{L}\mathbf{p}) - \mathbf{d}_\omega\|^2 + \frac{\varepsilon^2}{2}\|\mathbf{D}_\omega \mathbf{L}\mathbf{p}\|^2 \tag{11}$$

with $\mathbf{d}_\omega$ representing observed data in the frequency domain and $\mathbf{K}$ – wavefield sampling (recording) operator.

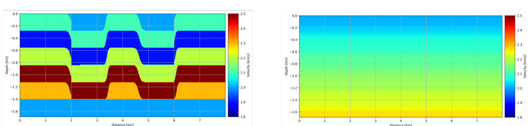
RESULTS

We evaluate the performance of the proposed algorithm on a synthetic example. To generate the data used for the inversion, we use the source wavelet in the Figure 1, the velocity model shown in the Figure 2a and the modeling operator in the Equation 6. The data contains only reflections, no energy below 3 Hz and maximum offset of 8 km. We start the inversion using the velocity model shown in the Figure 2b.



(a) (b)

Figure 1: Source wavelet used to model the observed data: (a) – in time, (b) – in the frequency domain.



(a) (b)

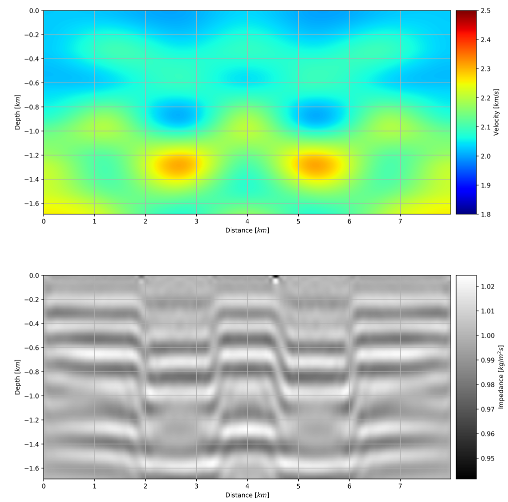
Figure 2: Velocity model used to model the observed data (a) and starting model for extended FWI (b) constant along the frequency axis.

We perform the inversion using a multi-scale approach both in space and in the frequency domain by using slowness-impedance parametrization (Equation 7). We minimize the objective function in the Equation 11 using non-linear conjugate gradient method (Fletcher and Reeves, 1964). The spline grid spacing for slowness model used in the inversion is refined at different stages as listed below:

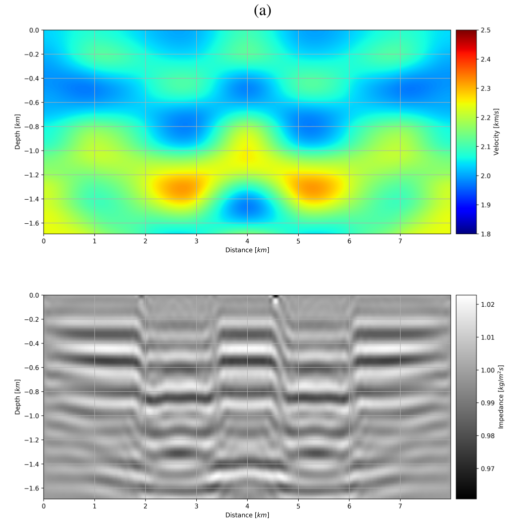
- (g_1) 1100 m x 190 m x 0.3Hz and 4-6 Hz data
- (g_2) 890 m x 150 m x 0.4Hz and 6-8 Hz data
- (g_3) 720 m x 110 m x 0.6Hz and 8-10 Hz data

In order to promote low-wavenumber updates to the slowness model, we start the inversion with the low frequency range of 4-6 Hz and coarse spline sampling (g_1). We then increase the data frequency range to 6-8 Hz and 8-10 Hz and refine the spline grid to (g_2) and (g_3) respectively. The inverted slowness model at a current spline grid is used as a starting model for the next grid refinement. The impedance model is updated at each iteration simultaneously with slowness and a constant homogeneous model is used as a starting impedance model at each of the $(g_1 - g_3)$ stages.

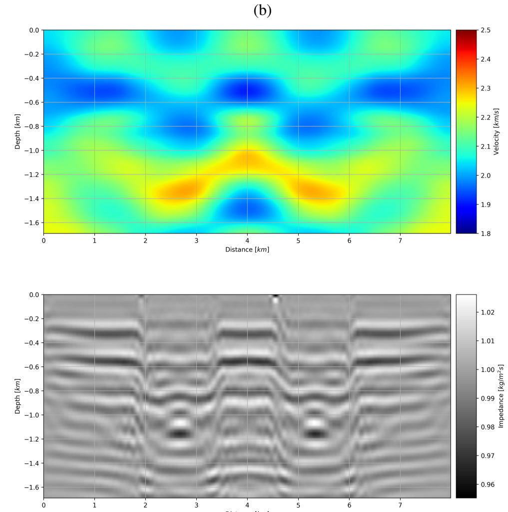
In order to get the "physical" models we average extended models across frequency axis that corresponds to the conventional imaging condition in the frequency domain. Figure 3



(a)



(b)

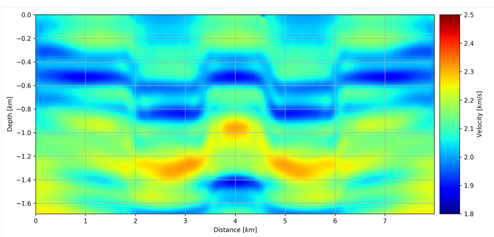


(c)

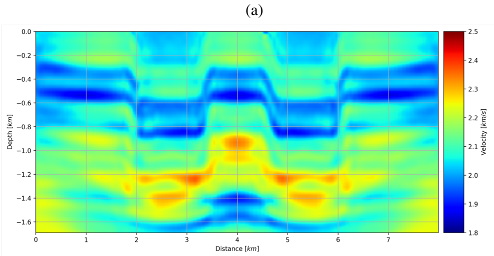
Figure 3: Inverted models at successive multi-scale steps averaged across frequency axis (zero time-lag): (a) – g_1, (b) – g_2, (c) – g_3; top – velocity, bottom – impedance model.

shows the inverted models after each stage of the spline grid refinement. As expected the impedance models possess high-wavenumber (reflectivity) features, while the velocity models are relatively smooth with gradually increased resolution going from grid (g_1) to (g_3). In all three spline grid refinement stages, both the data fitting and the regularization term demonstrate good convergence (Figure 5), and, as a result, accurate velocity models are recovered with almost no variation along the frequency axis.

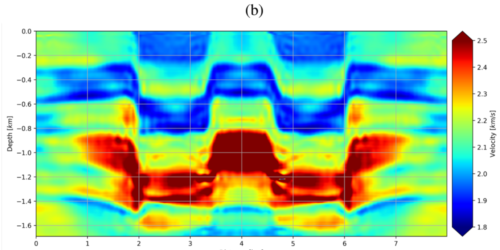
Next, to populate the model with higher wavenumbers, we perform extended FWI using slowness-only parametrization (Equation 6) and the starting velocity model in the Figure 3c. We start the inversion with the data in the 4-6 Hz frequency range (Figure 4a), followed by the extended FWI in the 6-8 Hz frequency range (Figure 4b). Finally, we run the conventional FWI using the full frequency range and reconstruct the velocity model shown in the Figure 4c. The edges of the reconstructed model are not reliable due to insufficient angular coverage. Additionally, some "ringing" artefacts are present because of the relatively narrow frequency bandwidths used in the previous steps. However, the main features of the true model (Figure 2a) and the velocity values are accurately recovered.



(a)



(b)

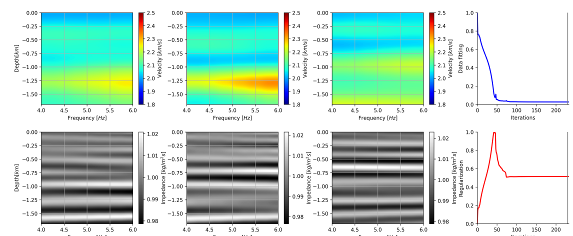


(c)

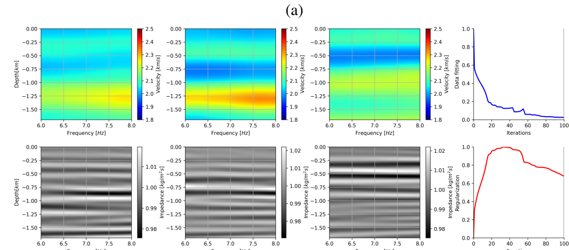
Figure 4: Inverted models (a-b) using extended slowness-only parametrization at successive multi-scale steps averaged across frequency axis (zero time-lag): (a) – 4-6Hz, (b) – 6-8Hz. Final inverted velocity model after FWI (c).

CONCLUSION

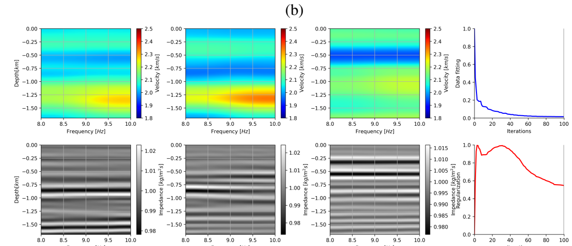
We propose a computationally efficient frequency-domain formulation of full-waveform inversion using extended slowness and impedance models. With this method, wavefields are propagated using a frequency-variant slowness, facilitating non-physical wave propagation and scattering and enabling accurate modeling of data residuals even when starting from an inaccurate velocity model. Our method does not involve solving an intermediate least-squares problem for extended reflectivity, enhancing computational efficiency. We have implemented a multi-scale approach in space by employing variable spline grid parametrization, enabling the recovery of accurate slowness models from rough starting estimates and data lacking low-frequency components. Our results on a synthetic example demonstrate the effectiveness of the proposed method. Through successive iterations of multi-scale refinement, we have achieved accurate velocity model reconstruction.



(a)



(b)



(c)

Figure 5: Inverted models at successive multi-scale steps as function of frequency: (a) – g_1, (b) – g_2, (c) – g_3; left – at $x = 2$ km, middle – at $x = 5$ km, right – at $x = 7$ km; top – velocity, bottom – impedance model. Blue curve shows the normalized values of the data fitting term, red curve – regularization term of the objective function.